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Local scale invariances in the bosonic contact and pair-contact processes

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Abstract
Local scale invariance for ageing systems without detailed balance is tested through studying the dynamical symmetries of the critical bosonic contact process and the critical bosonic pair-contact process. Their field-theoretical actions can be split into a Schrödinger-invariant term and a pure noise term. It is shown that the two-time response and correlation functions are reducible to certain multipoint response functions which depend only on the Schrödinger-invariant part of the action. For the bosonic contact process, the representation of the Schrödinger group can be derived from the free diffusion equation, whereas for the bosonic pair-contact process, a new representation of the Schrödinger group related to a nonlinear Schrödinger equation with dimensionful couplings is constructed. The resulting predictions of local scale invariance for the two-time responses and correlators are completely consistent with the exactly-known results in both models.

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1. Introduction

The concept of scale invariance is central to the modern understanding of critical phenomena in and out of equilibrium. Its exploitation through the renormalization group has in particular led to the recognition of universal critical exponents and scaling functions which describe the behaviour of physical observables (see, e.g., [1] and references therein). Here we are

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concerned with the slow dynamics of systems brought rapidly to their critical point and/or into a phase with more than one thermodynamically stable state. Such a kind of behaviour is typical for glassy systems but also occurs in simple magnets with a purely relaxational dynamics which were quenched from a disordered state to a final temperature $T \leqslant T_c$, where $T_c > 0$ is the critical temperature. For the latter, it is now understood that the dynamics is governed by a single time-dependent length scale $L = L(t) \sim t^{1/z}$ for t sufficiently large and where z is the dynamical exponent [2]. As an example, consider simple magnets relaxing towards equilibrium. For phase-ordering kinetics ($T < T_c$), the Bray–Rutenberg theory shows that dynamical scaling together with the assumption of a Porod law for the time-dependent structure factor predicts the value of z [3]; whereas for $T = T_c$ the value of z is computed from critical (equilibrium) dynamics. More recently, it has been understood that the study of two-time observables provides a further and deeper insight, in particular the ageing behaviour is made explicit through the breaking of time-translation invariance. The challenge is now to find the values of the associated non-equilibrium (ageing) exponents and also the form of the scaling functions, see below for the precise definitions.

A common way to study this problem is through a Langevin equation which should describe the dynamics of a coarse-grained order parameter. This may be turned into a field theory and renormalization-group methods then allow us to extract values of these exponents, in quite good but not perfect agreement with the results of direct numerical simulations [4, 5]. On the other hand, the resulting predictions for the scaling functions appear to be far from the numerical results, see [6, 7]. An alternative approach seeks to extend dynamical scaling to a larger group of 'local' scale transformations [8], see [9] for a recent review. In the framework of a de Dominicis–Janssen-type theory [10, 11], the effective action $S = S[\phi, \widetilde{\phi}]$ is given in terms of the order-parameter field ϕ and its associated response field $\widetilde{\phi}$. Furthermore, for systems in contact with a thermal bath such that detailed balance holds one always has the decomposition $S[\phi, \widetilde{\phi}] = S_0[\phi, \widetilde{\phi}] + S_b[\widetilde{\phi}]$ into a 'deterministic' part S_0 which can be derived from the Langevin equation when all noise terms are dropped and the 'noise' term $S_b[\widetilde{\phi}]$ which depends only on the response function [12]. The form of response functions can then be found from the requirement of covariance under the group of local scale transformations. As we shall explain below, correlation functions can be reduced to certain integrals of higher, multipoint response functions [12]. This approach yields the form of the scaling functions whereas the exponents are treated as parameters whose values have to be supplied[7] and reproduce perfectly the results of both analytical and simulational studies of many common spin systems undergoing phase-ordering kinetics where $z = 2$.[8]

It is an established fact that the basic Langevin equation for the order-parameter does not admit any symmetries beyond dilatations and (space-)translations (see [14] for a recent discussion). However, it has been shown that at least for simple magnets it is enough to concentrate on the dynamical symmetries of the *deterministic* part of the Langevin equation only, as given by the action S_0. In particular, Schrödinger invariance of that deterministic part is sufficient to be able to derive the two-time correlations $C(t, s)$ and two-time response functions $R(t, s)$ explicitly [12]. There is an exact agreement for systems such as the spherical model, the XY model in spin-wave approximation or the voter model which are all described

[7] This is close in spirit to the treatment of *equilibrium* phase transitions through conformal invariance, which fixes the form of the n-point correlators in terms of the scaling dimensions of the scaling fields [1]. Furthermore, those exponents can be determined exactly in 2D from symmetry considerations (i.e. representation theory of the Virasoro algebra) alone since the conformal symmetry is infinite-dimensional in that case.

[8] For non-equilibrium critical dynamics where $z \neq 2$ in general one also has a good match with numerical data for the response functions in direct space but systematic differences may appear in momentum-space calculations, e.g. in the $2D$ Ising model for $t/s \lesssim 10$ [13].

by a linear Langevin equation. A good agreement with simulations of Ising, Potts and XY models was found as well [7, 15, 16].

In this paper, we extend the treatment of local scale invariance to ageing systems with a dynamical exponent $z = 2$ but without detailed balance. Working with a de Dominicis–Janssen-type theory, we find again a decomposition $S[\phi, \widetilde{\phi}] = S_0[\phi, \widetilde{\phi}] + S_b[\phi, \widetilde{\phi}]$ into a 'deterministic', Schrödinger-invariant term S_0 and 'noise' terms, each of which contains at least one response field more than order-parameter fields (explicit expressions will be given in sections 2 and 3). Then the Bargman superselection rules which follow from the Galilei invariance of S_0 are enough to establish that again the two-time response function is noise independent and the two-time correlation function can be reduced to a finite sum of response functions the form of which is strongly constrained again by the requirement of their Schrödinger covariance. These developments provide further evidence for *a hidden non-trivial local scale invariance in ageing systems* which manifests itself directly in the 'deterministic' part (see [17] for the construction of Schrödinger-invariant semi-linear kinetic equations) but strongly constrains the full noisy correlations and responses.

We test the present framework of local scale invariance in two exactly solvable systems with a nonlinear coarse-grained Langevin equation. A convenient set of models with non-trivial ageing behaviour is furnished by the bosonic contact [18] and pair-contact processes [19], both at criticality. These systems are defined as follows. Consider a set of particles of a single species A which move on the sites of a hypercubic lattice in d dimensions. On any site one may have an arbitrary (non-negative) number of particles[9]. Single particles may hop to a nearest-neighbour site with unit rate and in addition, the following single-site creation and annihilation processes are admitted:

$$mA \xrightarrow{\mu} (m+1)A, \qquad pA \xrightarrow{\lambda} (p-\ell)A; \qquad \text{with rates } \mu \text{ and } \lambda, \tag{1}$$

where ℓ is a positive integer such that $|\ell| \leqslant p$. We are interested in the following special cases:

(i) *Critical bosonic contact process:* $p = m = 1$. Here only $\ell = 1$ is possible. Furthermore the creation and annihilation rates are set equal $\mu = \lambda$.

(ii) *Critical bosonic pair-contact process:* $p = m = 2$. We fix $\ell = 2$, set $2\lambda = \mu$ and define the control parameter[10]

$$\alpha := \frac{3\mu}{2D}. \tag{2}$$

The dynamics is described in terms of a master equation which may be written in a Hamiltonian form $\partial_t|P(t)\rangle = -H|P(t)\rangle$ where $|P(t)\rangle$ is the time-dependent state vector and the Hamiltonian H can be expressed in terms of creation and annihilation operators $a(\boldsymbol{x}, t)^\dagger$ and $a(\boldsymbol{x}, t)$ [20–22]. It is well known that these models are critical in the sense that their relaxation towards the steady state is algebraically slow [18, 19, 24]. In particular, the local particle density is $\rho(\boldsymbol{x}, t) := \langle a(\boldsymbol{x}, t)\rangle$. Its spatial average remains constant in time:

$$\int \mathrm{d}\boldsymbol{x}\, \rho(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{x} \langle a(\boldsymbol{x}, t)\rangle = \rho_0, \tag{3}$$

where ρ_0 is the initial mean particle density. We are interested in the two-time connected correlation function

$$G(\boldsymbol{r}; t, s) := \langle a(\boldsymbol{x}, t) a(\boldsymbol{x} + \boldsymbol{r}, s)\rangle - \rho_0^2 \tag{4}$$

[9] This property distinguishes the models at hand from the conventional ('fermionic') contact and pair-contact processes whose critical behaviour is completely different.

[10] If instead we would treat a coagulation process $2A \to A$, where $\ell = 1$, the results presented in the text are recovered by setting $\lambda = \mu$ and $\alpha = \mu/D$.

Table 1. Ageing exponents of the critical bosonic contact and pair-contact processes in the different regimes. The results for the bosonic contact process hold for an arbitrary dimension d, but for the bosonic pair-contact process they only apply if $d > 2$, since $\alpha_c = 0$ for $d \leqslant 2$.

		Bosonic pair-contact process	
	Bosonic contact process	$\alpha < \alpha_c$	$\alpha = \alpha_c$
a	$\frac{d}{2} - 1$	$\frac{d}{2} - 1$	$\frac{d}{2} - 1$
b	$\frac{d}{2} - 1$	$\frac{d}{2} - 1$	0 if $2 < d < 4$; $\frac{d}{2} - 2$ if $d > 4$

and take an uncorrelated initial state; hence $G(\boldsymbol{r}; 0, 0) = 0$. The linear two-time response function is found by adding a particle-creation term $\sum_{\boldsymbol{x}} h(\boldsymbol{x}, t)(a^\dagger(\boldsymbol{x}, t) - 1)$ to the quantum Hamiltonian H and taking the functional derivative

$$R(\boldsymbol{r}; t, s) := \left.\frac{\delta\langle a(\boldsymbol{r} + \boldsymbol{x}, t)\rangle}{\delta h(\boldsymbol{x}, s)}\right|_{h=0}. \tag{5}$$

We have previously analysed these quantities in the scaling limit where both t, s and $t - s$ become large with respect to some microscopic reference time. The results are as follows [24]: consider the autocorrelation and autoresponse functions, which satisfy the scaling forms

$$G(t, s) := G(\mathbf{0}; t, s) = s^{-b} f_G(t/s) \tag{6}$$

$$R(t, s) := R(\mathbf{0}; t, s) = s^{-1-a} f_R(t/s), \tag{7}$$

where the values of the exponents a and b are listed in table 1. Here the critical value α_c for the pair-contact process is explicitly given by [19]

$$\frac{1}{\alpha_c} = 2\int_0^\infty \mathrm{d}u\,(\mathrm{e}^{-4u} I_0(4u))^d, \tag{8}$$

where I_0 is a modified Bessel function. The dynamical behaviour of the contact process is independent of α. For the critical bosonic pair-contact process, there is a clustering transition between a spatially homogeneous state for $\alpha < \alpha_c$ and a highly inhomogeneous one for $\alpha > \alpha_c$ where dynamical scaling does not hold. These two transitions are separated by a multicritical point at $\alpha = \alpha_c$. Since our models do not satisfy detailed balance, there is no reason why the exponents a and b should coincide and our result $a \neq b$ for the bosonic pair-contact process is perfectly natural.

While the scaling function $f_R(y) = (y-1)^{-d/2}$ has a very simple form, the autocorrelator scaling function has an integral representation

$$f_G(y) = \mathcal{G}_0 \int_0^1 \mathrm{d}\theta\, \theta^{a-b}(y + 1 - 2\theta)^{-d/2}, \tag{9}$$

where the values for a and b are given in table 1 and $\mathcal{G}_0$ is a known normalization constant. The explicit scaling functions are listed up to normalization in table 2 [24]. In this paper, we shall study to what extent their form can be understood from local scale invariance.

This paper is organized as follows. In section 2 we treat the bosonic contact process in its field-theoretical formulation. The action is split into a Schrödinger-invariant term S_0 and a noise term S_b and we show how the response and correlation functions can be exactly reduced to certain noiseless three- and four-point response functions. In this reduction the Bargman superselection rules which follow from the Schrödinger invariance of S_0 play a central role. These tools allow us to predict the response- and correlation functions which will

Table 2. Scaling functions (up to normalization) of the autoresponse and autocorrelation of the critical bosonic contact and bosonic pair-contact processes.

			$f_R(y)$	$f_G(y)$
Contact process			$(y-1)^{-\frac{d}{2}}$	$(y-1)^{-\frac{d}{2}+1}-(y+1)^{-\frac{d}{2}+1}$
Pair contact process	$\alpha<\alpha_c$	$d>2$	$(y-1)^{-\frac{d}{2}}$	$(y-1)^{-\frac{d}{2}+1}-(y+1)^{-\frac{d}{2}+1}$
		$2<d<4$	$(y-1)^{-\frac{d}{2}}$	$(y+1)^{-\frac{d}{2}}{}_2F_1\left(\frac{d}{2},\frac{d}{2};\frac{d}{2}+1;\frac{2}{y+1}\right)$
	$\alpha=\alpha_c$	$d>4$	$(y-1)^{-\frac{d}{2}}$	$(y+1)^{-\frac{d}{2}+2}-(y-1)^{-\frac{d}{2}+2}+(d-4)(y-1)^{-\frac{d}{2}+1}$

be compared to the exact results of table 2. In section 3 the same programme is carried out for the bosonic pair-contact process but as we shall see, the Schrödinger-invariant term S_0 of its action is now related to a *non*linear Schrödinger equation. The treatment of this requires an extension of the usual representation of the Schrödinger Lie-algebra which now includes a dimensionful coupling constant. The construction is carried out in appendix A. The required n-point correlation functions coming from this new representation are derived in appendices B and C. Finally, in section 4 we conclude.

2. The contact process

2.1. Field-theoretical description

The master equation which describes the critical bosonic contact process as defined in section 1 can be turned into a field theory in a standard fashion through an operator formalism which uses a particle annihilation operator $a(\boldsymbol{r},t)$ and its conjugate $a^\dagger(\boldsymbol{r},t)$ (see, for instance, [20, 22] for a detailed discussion of the technique). Since we shall be interested in the connected correlator, we consider the shifted field and furthermore introduce the shifted response field:

$$\phi(\boldsymbol{r},t):=a(\boldsymbol{r},t)-\rho_0 \qquad \widetilde{\phi}(\boldsymbol{r},t):=\bar{a}(\boldsymbol{r},t)=a^\dagger(\boldsymbol{r},t)-1 \tag{10}$$

such that $\langle\phi(\boldsymbol{r},t)\rangle=0$ (our notation implies a mapping between operators and quantum fields, using the known equivalence between the operator formalism and the path-integral formulation [22, 23]). As we shall see, these fields ϕ and $\widetilde{\phi}$ will become the natural quasiprimary fields from the point of view of local scale invariance. We remark that the response function is not affected by this shift, since

$$R(\boldsymbol{r},\boldsymbol{r}';t,s)=\frac{\delta\langle a(\boldsymbol{r},t)\rangle}{\delta h(\boldsymbol{r}',s)}=\frac{\delta\langle\phi(\boldsymbol{r},t)\rangle}{\delta h(\boldsymbol{r}',s)}. \tag{11}$$

Then the field-theory action reads, where μ is the reaction rate [25],

$$\begin{aligned} S[\phi,\widetilde{\phi}] &= \int \mathrm{d}\boldsymbol{R}\int \mathrm{d}u[\widetilde{\phi}(2\mathcal{M}\partial_u-\nabla^2)\phi-\mu\widetilde{\phi}^2(\phi+\rho_0)] \\ &= S_0[\phi,\widetilde{\phi}]+S_b[\phi,\widetilde{\phi}]. \end{aligned} \tag{12}$$

To keep expressions shorter, we have suppressed the arguments of $\phi(\boldsymbol{R},u)$ and $\tilde{\phi}(\boldsymbol{R},u)$ under the integrals and we shall also do so often in what follows, if no ambiguity arises. The diffusion constant D is related to the 'mass' $\mathcal{M}$ through $D=(2\mathcal{M})^{-1}$. We have decomposed the action as follows:

$$S_0[\phi,\widetilde{\phi}]:=\int \mathrm{d}\boldsymbol{R}\int \mathrm{d}u[\widetilde{\phi}(2\mathcal{M}\partial_u-\nabla^2)\phi] \tag{13}$$

describes the deterministic[11], noiseless part whereas the noise is described by

$$S_b[\phi, \widetilde{\phi}] := -\mu \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u[\widetilde{\phi}^2(\phi + \rho_0)] \tag{14}$$

quite analogously to what happens in the kinetics of simple magnets (see [12] for details).

In principle, an initial correlator $G(\boldsymbol{r}; 0, 0)$ could be assumed and will lead to a further contribution S_{ini} to the action. For critical systems, one usually employs a term of the form $S_{\text{ini,st}} = -\frac{\tau_0}{2}\int \mathrm{d}\boldsymbol{R}(\phi(\boldsymbol{R}, 0) - \langle\phi(\boldsymbol{R}, 0)\rangle)^2$, see e.g. [5, 11] but this would have for us the disadvantage that it explicitly breaks Galilei invariance. We shall rather make use of the Galilei invariance of the noiseless action $S_0[\phi, \widetilde{\phi}]$ and use as an initial term [4, 12]

$$S_{\text{ini}}[\widetilde{\phi}] = -\frac{1}{2}\int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{R}'\widetilde{\phi}(\boldsymbol{R}, 0)G(\boldsymbol{R} - \boldsymbol{R}'; 0, 0)\widetilde{\phi}(\boldsymbol{R}', 0). \tag{15}$$

Because of the initial condition $G(\boldsymbol{R}; 0, 0) = 0$, however, $S_{\text{ini}}[\widetilde{\phi}] = 0$ and we shall not need to consider it any further.

From the action (12), n-point functions can then be computed as usual:

$$\langle\phi_1(\boldsymbol{r}_1, t_1)\ldots\phi_n(\boldsymbol{r}_n, t_n)\rangle = \int \mathcal{D}\phi\,\mathcal{D}\widetilde{\phi}\,\phi_1(\boldsymbol{r}_1, t_1)\ldots\phi_n(\boldsymbol{r}_n, t_n)\exp(-S[\phi, \widetilde{\phi}]), \tag{16}$$

which through the decomposition (12) can be written as an average of the noiseless theory

$$\langle\phi_1(\boldsymbol{r}_1, t_1)\ldots\phi_n(\boldsymbol{r}_n, t_n)\rangle = \langle\phi_1(\boldsymbol{r}_1, t_1)\ldots\phi_n(\boldsymbol{r}_n, t_n)\exp(-S_b[\phi, \widetilde{\phi}])\rangle_0, \tag{17}$$

where $\langle\ldots\rangle_0$ denotes the expectation value with respect to the noiseless theory.

2.2. *Symmetries of the noiseless theory*

In what follows, we shall need some symmetry properties of the noiseless part described by the action $S_0[\phi, \widetilde{\phi}]$ which we now briefly recall. The noiseless equation of motion for the field ϕ is a free diffusion equation $2\mathcal{M}\partial_t\phi(\boldsymbol{x}, t) = \nabla^2\phi(\boldsymbol{x}, t)$. Its dynamical symmetry group is the well-known Schrödinger group Sch(d) [26, 27] which acts on spacetime coordinates $(\boldsymbol{r}, t)$ as $(\boldsymbol{r}, t) \mapsto (\boldsymbol{r}', t') = g(\boldsymbol{r}, t)$ where

$$t \longrightarrow t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad \boldsymbol{r} \longrightarrow \boldsymbol{r}' = \frac{\mathcal{R}\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}}{\gamma t + \delta}; \qquad \alpha\delta - \beta\gamma = 1 \tag{18}$$

and where $\mathcal{R}$ is a rotation matrix. Solutions ϕ of the free diffusion equation are carried to other solutions of the same equation and ϕ transforms as

$$\phi(\boldsymbol{r}, t) \longrightarrow (T_g\phi)(\boldsymbol{r}, t) = f_g[g^{-1}(\boldsymbol{r}, t)]\phi[g^{-1}(\boldsymbol{r}, t)], \tag{19}$$

where the companion function f_g is known explicitly and contains the so-called mass $\mathcal{M} = (2D)^{-1}$ [27, 28]. We list the generators of the Lie algebra $\mathfrak{sch}_1 = \text{Lie(Sch(1))}$ in one spatial dimension [29] as follows:

$$\begin{aligned}
X_{-1} &= -\partial_t \\
X_0 &= -t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2} \\
X_1 &= -t^2\partial_t - tr\partial_r - xt - \frac{\mathcal{M}}{2}r^2 \\
Y_{-\frac{1}{2}} &= -\partial_r \\
Y_{\frac{1}{2}} &= -t\partial_r - \mathcal{M}r \\
M_0 &= -\mathcal{M}.
\end{aligned} \tag{20}$$

[11] This terminology is used since the equation of motion of ϕ following from S_0 is a partial differential equation and not a stochastic Langevin equation.

Table 3. Scaling dimensions and masses of some composite fields.

Field	Scaling dimension	Mass
ϕ	x	$\mathcal{M}$
$\widetilde{\phi}$	$\widetilde{x}$	$-\mathcal{M}$
$\widetilde{\phi}^2$	$\widetilde{x}_2$	$-2\mathcal{M}$
$\Upsilon := \widetilde{\phi}^2\phi$	x_Υ	$-\mathcal{M}$
$\Sigma := \widetilde{\phi}^3\phi$	x_Σ	$-2\mathcal{M}$
$\Gamma := \widetilde{\phi}^3\phi^2$	x_Γ	$-\mathcal{M}$

Fields transforming under Sch(d) are characterized by a scaling dimension and a mass. We list in table 3 some fields which we shall use below. We remark that for free fields one has

$$\widetilde{x}_2 = 2\widetilde{x}, \qquad x_\Upsilon = 2\widetilde{x} + x, \qquad x_\Sigma = 3\widetilde{x} + x, \qquad x_\Gamma = 3\widetilde{x} + 2x, \tag{21}$$

but these relations need no longer hold for interacting fields. On the other hand, from the Bargman superselection rules (see [30] and below) we expect that the masses of the composite fields as given in table 3 should remain valid for interacting fields as well.

Throughout this paper, we shall make the important assumption that the fields ϕ and $\widetilde{\phi}$ transform covariantly according to (19) under the Schrödinger group. By analogy with conformal invariance, such fields are called *quasiprimary* [8]. For quasiprimary fields the so-called Bargman superselection rules [30] hold true which state that

$$\langle \underbrace{\phi\ldots\phi}_{n}\underbrace{\widetilde{\phi}\ldots\widetilde{\phi}}_{m}\rangle_0 = 0 \qquad \text{unless} \quad n = m. \tag{22}$$

We recall the proof of these in appendix B. Before we consider the consequences of (22), we recall the well-known result in the form of noiseless n-point functions in ageing systems.

Since in ageing phenomena, time-translation invariance is broken, we must consider the subalgebra $\mathfrak{age}_1 \subset \mathfrak{sch}_1$ obtained by leaving out the generator of time translations X_{-1} [31]. Then the n-point function of quasiprimary fields ϕ_i, $i = 1, \ldots, n$, has to satisfy the covariance conditions [8, 29]

$$\left(\sum_{i=1}^{n} X_k^{(i)}\right)\langle\varphi_1(\boldsymbol{r}_1, t_1)\ldots\varphi_n(\boldsymbol{r}_n, t_n)\rangle_0 = 0 \qquad k \in \{0, 1\} \tag{23}$$

$$\left(\sum_{i=1}^{n} Y_m^{(i)}\right)\langle\varphi_1(\boldsymbol{r}_1, t_1)\ldots\varphi_n(\boldsymbol{r}_n, t_n)\rangle_0 = 0 \qquad m \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \tag{24}$$

where φ_i stands either for a quasiprimary field ϕ_i or a quasiprimary response field $\widetilde{\phi}_i$. The φ_i are characterized by their scaling dimension x_i and their mass $\mathcal{M}_i$. The generators X_k are then the extension of (20) to n-body operators and the superscript (i) refers to φ_i. The n-point function is zero unless the sum of all masses vanishes:

$$\sum_{i=1}^{n} \mathcal{M}_i = 0, \tag{25}$$

which reproduces the Bargman superselection rule (22). It is well known [8, 29] that the noiseless two-point function $R_0(\boldsymbol{r}, \boldsymbol{r}'; t, s) = \langle\varphi_1(\boldsymbol{r}, t)\varphi_2(\boldsymbol{r}, s)\rangle_0$ is completely determined by equations (23) and (24) up to a normalization constant:

$$R_0(\boldsymbol{r}, \boldsymbol{r}'; t, s) = R_0(t, s)\exp\left(-\frac{\mathcal{M}_1}{2}\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{(t-s)}\right)\delta(\mathcal{M}_1 + \mathcal{M}_2), \tag{26}$$

where the autoresponse function is given by

$$R_0(t,s) = r_0(t-s)^{-\frac{1}{2}(x_1+x_2)}\left(\frac{t}{s}\right)^{-\frac{1}{2}(x_1-x_2)}. \tag{27}$$

This reproduces the expected scaling form (7) together with the scaling function $f_R(y)$ as given in table 2 if we identify

$$x = x_1 = x_2 \qquad \text{and} \qquad x = a+1. \tag{28}$$

For the critical bosonic contact process, we read off from table 1 that $a = \frac{d}{2} - 1$. Hence one recovers $x = \frac{d}{2}$, as expected for a free-field theory.

2.3. Reduction formulae

We now show that the Bargman superselection rule (22) implies a reduction of the n-point function of the full theory to certain correlators of the noiseless theory, which is described by S_0 only. This can be done generalizing the arguments of [12].

First, for the computation of the response function, we add the term $\int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u\, \widetilde{\phi}(\boldsymbol{R},u)h(\boldsymbol{R},u)$ to the action. As usual the response function is

$$\begin{aligned} R(\boldsymbol{r},\boldsymbol{r}';t,s) &= \langle \phi(\boldsymbol{r},t)\widetilde{\phi}(\boldsymbol{r}',s)\rangle \\ &= \left\langle \phi(\boldsymbol{r},t)\widetilde{\phi}(\boldsymbol{r}',s)\exp\left(-\mu\int \mathrm{d}\boldsymbol{R}\int \mathrm{d}u\,\widetilde{\phi}^2(\boldsymbol{R},u)(\phi(\boldsymbol{R},u)+\rho_0)\right)\right\rangle_0 \\ &= \langle \phi(\boldsymbol{r},t)\widetilde{\phi}(\boldsymbol{r}',s)\rangle_0 = R_0(\boldsymbol{r},\boldsymbol{r}';t,s), \end{aligned} \tag{29}$$

where we expanded the exponential and applied the Bargman superselection rule. Indeed, the two-time response is just given by the response of the (Gaussian) noise-less theory. We have therefore reproduced the exact result of table 2 for the response function of the critical bosonic contact process.

Second, we have for the correlator

$$\begin{aligned} G(\boldsymbol{r},\boldsymbol{r}',t,s) = \Big\langle \phi(\boldsymbol{r},t)\phi(\boldsymbol{r}'s)\exp\left(-\mu\int \mathrm{d}\boldsymbol{R}\int \mathrm{d}u\,\widetilde{\phi}^2(\boldsymbol{R},u)\phi(\boldsymbol{R},u)\right) \\ \times \exp\left(-\mu\rho_0\int \mathrm{d}\boldsymbol{R}\int \mathrm{d}u\,\widetilde{\phi}^2(\boldsymbol{R},u)\right)\Big\rangle_0. \end{aligned} \tag{30}$$

Expanding both exponentials

$$\exp\left(-\mu\int \mathrm{d}\boldsymbol{R}\int \mathrm{d}u\,\widetilde{\phi}^2(\boldsymbol{R},u)\phi(\boldsymbol{R},u)\right) = \sum_{n=0}^{\infty}\frac{(-\mu)^n}{n!}\left(\int \mathrm{d}\boldsymbol{R}\int \mathrm{d}u\,\widetilde{\phi}^2(\boldsymbol{R},u)\phi(\boldsymbol{R},u)\right)^n$$

$$\exp\left(-\mu\rho_0\int \mathrm{d}\boldsymbol{R}\int \mathrm{d}u\,\widetilde{\phi}^2(\boldsymbol{R},u)\right) = \sum_{m=0}^{\infty}\frac{(-\rho_0\mu)^m}{m!}\left(\int \mathrm{d}\boldsymbol{R}\int \mathrm{d}u\,\widetilde{\phi}^2(\boldsymbol{R},u)\right)^m$$

and using the Bargman superselection rule (22), non-vanishing terms only arise if $2n + 2m = n + 2$ or else

$$n + 2m = 2. \tag{31}$$

This can only be satisfied for $n = 0$ and $m = 1$ or for $n = 2$ and $m = 0$. Hence the full noisy correlator is the sum of only two terms:

$$G(\boldsymbol{r},\boldsymbol{r}';t,s) = G_1(\boldsymbol{r},\boldsymbol{r}';t,s) + G_2(\boldsymbol{r},\boldsymbol{r}';t,s), \tag{32}$$

where the first contribution involves a three-point function of the composite field $\widetilde{\phi}^2$ of scaling dimension $\tilde{x}_2$ (see table 3):

$$G_1(\boldsymbol{r},\boldsymbol{r}';t,s) = -\mu\rho_0 \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u \langle \phi(\boldsymbol{r},t)\phi(\boldsymbol{r}',s)\widetilde{\phi}^2(\boldsymbol{R},u)\rangle_0, \tag{33}$$

whereas the second contribution comes from a four-point function and involves the composite field Υ (see table 3):

$$G_2(\boldsymbol{r},\boldsymbol{r}';t,s) = \frac{\mu^2}{2} \int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{R}' \int \mathrm{d}u\,\mathrm{d}u' \langle \phi(\boldsymbol{r},t)\phi(\boldsymbol{r}',s)\Upsilon(\boldsymbol{R},u)\Upsilon(\boldsymbol{R}',u')\rangle_0. \tag{34}$$

We see that the connected correlator is determined by three- and four-point functions of the noiseless theory. We now use the symmetries of that noiseless theory to determine the two-, three- and four-point functions as far as possible.

2.4. Correlator with noise

We consider $G_1(\boldsymbol{r},\boldsymbol{r}',t,s)$ first. The appropriate three-point function is given in appendix B, equation (B.24):

$$\begin{aligned}\langle \phi(\boldsymbol{r},t)\phi(\boldsymbol{r}',s)\widetilde{\phi}^2(\boldsymbol{R},u)\rangle_0 &= (t-s)^{x-\frac{1}{2}\tilde{x}_2}(t-u)^{-\frac{1}{2}\tilde{x}_2}(s-u)^{-\frac{1}{2}\tilde{x}_2} \\ &\quad\times \exp\left(-\frac{\mathcal{M}}{2}\frac{(\boldsymbol{r}-\boldsymbol{R})^2}{t-u} - \frac{\mathcal{M}}{2}\frac{(\boldsymbol{r}'-\boldsymbol{R})^2}{s-u}\right)\Psi_3(u_1,v_1)\Theta(t-u)\Theta(s-u)\end{aligned} \tag{35}$$

with

$$\begin{aligned} u_1 &= \frac{u}{t}\cdot\frac{[(s-u)(\boldsymbol{r}-\boldsymbol{R})-(t-u)(\boldsymbol{r}'-\boldsymbol{R})]^2}{(t-u)(s-u)^2} \\ v_1 &= \frac{u}{s}\cdot\frac{[(s-u)(\boldsymbol{r}-\boldsymbol{R})-(t-u)(\boldsymbol{r}'-\boldsymbol{R})]^2}{(t-u)^2(s-u)} \end{aligned} \tag{36}$$

and an undetermined scaling function Ψ_3. The Θ-functions have been introduced by hand because of causality but this could be justified through a more elaborate argument along the lines of [31]. Introduced into (33), this gives the general form for the contribution $G_1(\boldsymbol{r},\boldsymbol{r}';t,s)$. We concentrate here on the autocorrelator, i.e. $\boldsymbol{r}=\boldsymbol{r}'$ and find, with $y=t/s$,

$$\begin{aligned} G_1(t,s) &= -\mu\rho_0\, s^{-x-\frac{1}{2}\tilde{x}_2+\frac{d}{2}+1}\cdot(y-1)^{-(x-\frac{1}{2}\tilde{x}_2)} \\ &\quad\times \int_0^1 \mathrm{d}\theta\,(y-\theta)^{-\frac{1}{2}\tilde{x}_2}(1-\theta)^{-\frac{1}{2}\tilde{x}_2}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{R}\exp\left(-\frac{\mathcal{M}}{2}\boldsymbol{R}^2\frac{y+1-2\theta}{(y-\theta)(1-\theta)}\right) \\ &\quad\times H\left(\frac{\theta}{y}\frac{\boldsymbol{R}^2(y-1)^2}{(y-\theta)(1-\theta)^2},\theta\frac{\boldsymbol{R}^2(y-1)^2}{(y-\theta)^2(1-\theta)}\right), \end{aligned} \tag{37}$$

where H is an undetermined scaling function. Very much in the same way, we find for $G_2(t,s)$:

$$\begin{aligned} G_2(t,s) &= \frac{\mu^2}{2}s^{-x-x_\Upsilon+d+2}(y-1)^{-(x-x_\Upsilon)}\int_0^1\mathrm{d}\theta\int_0^1\mathrm{d}\theta'(y-\theta)^{-\frac{1}{2}x_\Upsilon}(1-\theta)^{-\frac{1}{2}x_\Upsilon} \\ &\quad\times(y-\theta')^{-\frac{1}{2}x_\Upsilon}(1-\theta')^{-\frac{1}{2}x_\Upsilon}\int_{\mathbb{R}^{2d}}\mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{R}'\exp\left(-\frac{\mathcal{M}}{2}\frac{\boldsymbol{R}^2}{1-\theta}-\frac{\mathcal{M}}{2}\frac{\boldsymbol{R}'^2}{1-\theta'}\right) \\ &\quad\times\Psi_4(\tilde{u}_3(\boldsymbol{R},\theta,\boldsymbol{R}',\theta'),\tilde{u}_4(\boldsymbol{R},\theta,\boldsymbol{R}',\theta'),\tilde{v}_3(\boldsymbol{R},\theta,\boldsymbol{R}',\theta'),\tilde{v}_4(\boldsymbol{R},\theta,\boldsymbol{R}',\theta')), \end{aligned} \tag{38}$$

where Ψ_4 is another undetermined function and the functions $\tilde{u}_3, \tilde{u}_4, \tilde{v}_3, \tilde{v}_4$ can be worked out from the appropriate expressions (B.26) in appendix B by the replacements $\mathbf{r}_3-\mathbf{r}_2\to\mathbf{R}$, $\mathbf{r}_4-\mathbf{r}_2\to\mathbf{R}'$, $t_2\to 1$, $t_1\to y$, $t_3\to\theta$, $t_4\to\theta'$ (remember that $\boldsymbol{r}_1=\boldsymbol{r}_2$).

As we have a free-field theory for the critical bosonic contact process, we expect from table 1 and equation (28) that $x = \widetilde{x} = d/2$ and hence the following scaling dimensions for the composite fields:

$$\tilde{x}_2 = d, \qquad x_\Upsilon = \tfrac{3}{2}d. \tag{39}$$

Consequently, the autocorrelator takes the general form

$$G(t,s) = s^{1-d/2} g_1(t/s) + s^{2-d} g_2(t/s). \tag{40}$$

For d larger than the lower critical dimension $d_* = 2$, the second term merely furnishes a finite-time correction. On the other hand, for $d < d_* = 2$, it would be the dominant one and we can only achieve agreement with the known exact result if we assume $\Psi_4 = 0$. In what follows, we shall discard the scaling function g_2 and shall concentrate on showing that our expressions for g_1 are compatible with the exact results given in table 2.

In order to do so, we choose the following special form for the function Ψ_3:

$$\Psi_3(u_1, v_1) = \Xi\left(\frac{1}{u_1} - \frac{1}{v_1}\right), \tag{41}$$

where Ξ remains an arbitrary function. Then we are back in the case already treated in [12]. We find

$$G_1(t,s) = -\mu\rho_0\, s^{\frac{d}{2}+1-x-\frac{1}{2}\tilde{x}_2}(y-1)^{\frac{1}{2}\tilde{x}_2-x-\frac{d}{2}} \int_0^1 \mathrm{d}\theta [(y-\theta)(1-\theta)]^{\frac{d}{2}-\frac{1}{2}\tilde{x}_2} \phi_1\left(\frac{y+1-2\theta}{y-1}\right), \tag{42}$$

where the function ϕ_1 is defined by

$$\phi_1(w) = \int \mathrm{d}\boldsymbol{R} \exp\left(-\frac{\mathcal{M}w}{2}\boldsymbol{R}^2\right) \Xi(\boldsymbol{R}^2). \tag{43}$$

As in [12] we choose

$$\phi_1(w) = \phi_{0,\mathrm{c}} w^{-1-a}. \tag{44}$$

This form for $\phi_1(w)$ guarantees that the three-point response function $\langle \phi(\boldsymbol{r},t)\phi(\boldsymbol{r},s)\phi^2(\boldsymbol{r}',u)\rangle_0$ is nonsingular for $t = s$. We have thus

$$G(t,s) = G_1(t,s) = s^{-b} f_G\left(\frac{t}{s}\right) \tag{45}$$

with

$$\begin{aligned} f_G(y) &= -\mu\rho_0\,\phi_{0,\mathrm{c}} \int_0^1 \mathrm{d}\theta (y+1-2\theta)^{-\frac{d}{2}} \\ &= \frac{2\mu\rho_0\,\phi_{0,\mathrm{c}}}{d}\left((y-1)^{-\frac{d}{2}+1} - (y+1)^{-\frac{d}{2}+1}\right) \end{aligned} \tag{46}$$

and we have reproduced the corresponding entry in table 2 for the critical bosonic contact process[12].

[12] We remark that for $2 < d < 4$, the same form of the autocorrelation function is also found in the critical voter model [32].

3. The pair-contact process

3.1. Field-theoretical description and reduction formula

For the pair-contact process we have two different cases, namely the case $\alpha < \alpha_c$ and the case at criticality $\alpha = \alpha_c$. The following considerations apply to both cases and we shall for the moment leave the value of α arbitrary and only fix it at a later state.

The action for the pair-contact process on the critical line is [25, equation (30)]

$$S[a, \bar{a}] = \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u[\bar{a}(2\mathcal{M}\partial_t - \nabla^2)a - \alpha\bar{a}^2 a^2 - \mu\bar{a}^3 a^2]. \tag{47}$$

As before, see equation (10), we switch to the quasiprimary fields $\phi(\boldsymbol{r}, t) = a(\boldsymbol{r}, t) - \rho_0$ and $\widetilde{\phi}(\boldsymbol{r}, t) = \bar{a}(\boldsymbol{r}, t)$. Then the action becomes

$$\begin{aligned} S[\phi, \widetilde{\phi}] = \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u\big[&\widetilde{\phi}(2\mathcal{M}\partial_t - \nabla^2)\phi - \alpha\widetilde{\phi}^2\phi^2 - \alpha\rho_0^2\widetilde{\phi}^2 \\ &- 2\alpha\rho_0\widetilde{\phi}^2\phi - \mu\widetilde{\phi}^3\phi^2 - 2\mu\rho_0\widetilde{\phi}^3\phi - \mu\rho_0^2\widetilde{\phi}^3\big] \\ = S_0[\phi, \widetilde{\phi}] &+ S_b[\phi, \widetilde{\phi}]. \end{aligned} \tag{48}$$

Also in this model, similarly to the treatment of section 2, a decomposition of the action into a first term with a non-trivial dynamic symmetry and a remaining noise term is sought such that the correlators and responses can be reexpressed in terms of certain n-point functions which only depend on S_0. The first term reads

$$S_0[\phi, \widetilde{\phi}] := \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}t[\widetilde{\phi}(2\mathcal{M}\partial_t - \nabla^2)\phi - \alpha\widetilde{\phi}^2\phi^2] \tag{49}$$

and we derive its Schrödinger invariance in appendix A. The remaining part is the noise term which reads

$$S_b[\phi, \widetilde{\phi}] = \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u\big[-\alpha\rho_0^2\widetilde{\phi}^2 - 2\alpha\rho_0\widetilde{\phi}^2\phi - \mu\widetilde{\phi}^3\phi^2 - 2\mu\rho_0\widetilde{\phi}^3\phi - \rho_0^2\widetilde{\phi}^3\big]. \tag{50}$$

Also in this case the Bargman superselection rule (22) holds true. This means that we can proceed now in a very similar way as before[13]. First we have to check which n-point functions contribute to the response and correlation function. We rewrite $\exp(-S_b[\phi, \widetilde{\phi}])$ as a product of five exponentials and expand each factor. The indices of the sums are denoted by k_i for the ith term in (50), for instance for the first term

$$\exp\left(-\int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u\, \alpha\rho_0^2\widetilde{\phi}^2(\boldsymbol{R}, u)\right) = \sum_{k_1=0}^{\infty} \frac{1}{k_1!}\left(-\int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u\, \alpha\rho_0^2\widetilde{\phi}^2(\boldsymbol{R}, u)\right)^{k_1}. \tag{51}$$

For the response function again only the first term of each sum contributes, that is,

$$R(\boldsymbol{r}, \boldsymbol{r}'; t, s) = R_0(\boldsymbol{r}, \boldsymbol{r}'; t, s) \tag{52}$$

is noise independent. For the correlation function, we have the condition $2k_1 + 2k_2 + 3k_3 + 3k_4 + 3k_5 = 2 + k_2 + 2k_3 + k_4$ or simply

$$2k_1 + k_2 + k_3 + 2k_4 + 3k_5 = 2, \tag{53}$$

which implies immediately that

$$k_5 = 0. \tag{54}$$

[13] This argument works provided each term in S_b contains at least one response field $\widetilde{\phi}$ more than order-parameter fields ϕ.

Table 4. Contributions to the correlation function: the first column shows how we denote the contribution, the next four columns give the value of the corresponding indices. The sixth column lists the composite field(s) involved, the seventh column how we denote the scaling dimension of that field. The last column lists whether it is a three- or four-point function that contributes.

Contribution	k_1	k_2	k_3	k_4	Composite field	Scaling dimension	3-point/4-point
$G_1(t,s)$	1	0	0	0	$\widetilde{\phi}^2$	$\tilde{x}_2$	3-point
$G_2(t,s)$	0	2	0	0	Υ	x_Υ	4-point
$G_3(t,s)$	0	0	2	0	Γ	x_Γ	4-point
$G_4(t,s)$	0	0	0	1	Σ	x_Σ	3-point
$G_5(t,s)$	0	1	1	0	Υ and Γ	x_Υ, x_Γ	4-point

In table 4 we list the five different contributions to the correlation function. We denote also the form of the composite field, its scaling dimension and whether it is a three- or four-point function which contributes. A short inspection of the general form of the n-points function given in appendix B shows that the contributions have the form (with $y = t/s$)

$$G_1(t,s) = s^{-x-\frac{1}{2}\tilde{x}_2+\frac{d}{2}+1} f_1(y), \qquad G_4(t,s) = s^{-x-\frac{1}{2}x_\Sigma+\frac{d}{2}+1} f_4(y) \tag{55}$$

for the three-point functions and

$$G_2(t,s) = s^{-x-x_\Upsilon+d+2} f_2(y), \qquad G_3(t,s) = s^{-x-x_\Gamma+d+2} f_3(y)$$
$$G_5(t,s) = s^{-x-\frac{1}{2}x_\Upsilon-\frac{1}{2}x_\Gamma+d+2} f_5(y)$$

for the four-point functions. The scaling functions $f_i(y)$ involve arbitrary functions $\tilde{\Psi}_i$ which are not fixed by the symmetries (see appendix B for details). As we do not have a free-field theory in this case we can not make any assumptions about the value of the scaling dimensions of the composite fields. Therefore we do not know which terms will be the leading ones in the scaling regime. However, it turns out that the term $G_1(t,s)$ alone can reproduce our result correctly. Thus we set the scaling functions $f_n = 0$ with $n = 2, \ldots, 5$, analogously to the last section. We now concentrate on

$$G_1(t,s) = \alpha\rho_0^2 \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}u \langle \phi(\boldsymbol{r},t)\phi(\boldsymbol{r},s)\widetilde{\phi}^2(\boldsymbol{R},u)\rangle_0. \tag{56}$$

3.2. Symmetries of the noiseless theory

As in the last chapter, we require for the calculation of the two- and three-point functions the symmetries of the following nonlinear Schrödinger equation obtained from (49),

$$2\mathcal{M}\partial_t\phi(\boldsymbol{x},t) = \nabla^2\phi(\boldsymbol{x},t) + \mathcal{F}(\phi,\widetilde{\phi}) \tag{57}$$

with a nonlinear potential

$$\mathcal{F}(\phi,\widetilde{\phi}) = -g\phi^2(\boldsymbol{x},t)\widetilde{\phi}(\boldsymbol{x},t). \tag{58}$$

While for a constant g the symmetries of this equation are well known, it was pointed out recently that g rather should be considered as a dimensionful quantity and hence should transform under local scale transformations as well [17]. This requires an extension of the generators used so far and we shall give this in appendix A. The computation of the n-point functions covariant with respect to these new generators is given in appendices B and C. In doing so, we have for technical simplicity assumed that to each field φ_i there is one associated coupling constant g_i and only at the end, we let

$$g_1 = \cdots = g_n =: g. \tag{59}$$

Therefore, from equation (52) we find for the response function (see (C.12))

$$R_0(\boldsymbol{r}, \boldsymbol{r}'; t, s) = (t-s)^{-\frac{1}{2}(x_1+x_2)} \left(\frac{t}{s}\right)^{-\frac{1}{2}(x_1-x_2)} \exp\left(-\frac{\mathcal{M}}{2}\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{t-s}\right) \tilde{\Psi}_2\left(\frac{t}{s}\frac{t-s}{g^{1/y}}, \frac{g}{(t-s)^y}\right) \tag{60}$$

with an undetermined scaling function $\tilde{\Psi}_2$. This form is clearly consistent with our results in table 2 if we identify

$$x := x_1 = x_2 = a + 1 = \frac{d}{2}, \qquad \tilde{\Psi}_2 = \text{const.} \tag{61}$$

This holds true for both $\alpha < \alpha_c$ and $\alpha = \alpha_c$. In distinction with the bosonic contact process, the symmetries of the noiseless part S_0 do not fix the response function completely but leave a certain degree of flexibility in the form of the scaling function $\tilde{\Psi}_2$.

For the calculation of the correlator we need from equation (56) the following three-point function:

$$\langle \phi(\boldsymbol{r}, t)\phi(\boldsymbol{r}', s)\tilde{\phi}^2(\boldsymbol{R}, u)\rangle_0 = (t-s)^{x-\frac{1}{2}\tilde{x}_2}(t-u)^{-\frac{1}{2}\tilde{x}_2}(s-u)^{-\frac{1}{2}\tilde{x}_2}$$
$$\times \exp\left(-\frac{\mathcal{M}}{2}\frac{(\boldsymbol{r}-\boldsymbol{R})^2}{t-u} - \frac{\mathcal{M}}{2}\frac{(\boldsymbol{r}'-\boldsymbol{R})^2}{s-u}\right) \tilde{\Psi}_3(u_1, v_1, \beta_1, \beta_2, \beta_3) \tag{62}$$

with

$$u_1 = \frac{u}{t}\frac{[(s-u)(\boldsymbol{r}-\boldsymbol{R}) - (t-u)(\boldsymbol{r}'-\boldsymbol{R})]^2}{(t-u)(s-u)^2} \tag{63}$$

$$v_1 = \frac{u}{s}\frac{[(s-u)(\boldsymbol{r}-\boldsymbol{R}) - (t-u)(\boldsymbol{r}'-\boldsymbol{R})]^2}{(t-u)^2(s-u)} \tag{64}$$

$$\beta_1 = \frac{1}{s_2}\frac{\alpha^{1/y}}{(t-u)^2}, \qquad \beta_2 = \frac{1}{s_2}\frac{\alpha^{1/y}}{(s-u)^2}, \qquad \beta_3 = \alpha^{1/y}s_2 \tag{65}$$

$$s_2 = \frac{1}{t-u} + \frac{1}{u}. \tag{66}$$

We choose the following realization for $\tilde{\Psi}_3$:

$$\tilde{\Psi}_3(u_1, v_1, \beta_1, \beta_2, \beta_3) = \Xi\left(\frac{1}{u_1} - \frac{1}{v_1}\right)\left[-\frac{(\sqrt{\beta_1}-\sqrt{\beta_2})\sqrt{\beta_3}}{\beta_3 - \sqrt{\beta_2\beta_3}}\right]^{(a-b)}, \tag{67}$$

where the scaling function Ξ was already encountered in equation (41) for the bosonic contact process. We now have to distinguish the two different cases $\alpha < \alpha_c$ and $\alpha = \alpha_c$. For the first case $\alpha < \alpha_c$, we have $a - b = 0$ so that the last factor in (67) disappears and we simply return to the expressions already found for the bosonic contact process, in agreement with the known exact results. However, at the multicritical point $\alpha = \alpha_c$ we have $a - b \neq 0$ and the last factor becomes important. We point out that only the presence or absence of this factor distinguishes the cases $\alpha < \alpha_c$ and $\alpha = \alpha_c$.

If we substitute the values for β_1, β_2 and β_3, $\tilde{\Psi}_3$ becomes

$$\tilde{\Psi}_3(u_1, v_1, \beta_1, \beta_2, \beta_3) = \Xi\left(\frac{1}{u_1} - \frac{1}{v_1}\right)\left[\frac{\theta(y-1)}{(y-\theta)(1-\theta)}\right]^{(a-b)}. \tag{68}$$

This factor does not involve $\boldsymbol{R}$ so that we obtain in a similar way as before

$$G_1(t,s) = s^{-b}(y-1)^{(b-a)-a-1}\int_0^1 \mathrm{d}\theta[(y-\theta)(1-\theta)]^{a-b} \times \phi_1\left(\frac{y+1-2\theta}{y-1}\right)\left[\frac{\theta(y-1)}{(y-\theta)(1-\theta)}\right]^{a-b}, \quad (69)$$

where we have identified

$$\tilde{x}_2 = 2(b-a)+d. \quad (70)$$

$G_1(t,s)$ reduces to expression (9) if we choose the same expression for $\phi_1(w)$ as before. We have thus reproduced all scaling functions correctly.

4. Conclusions

The objective of our investigation has been to test further the recent proposal of using the non-trivial dynamical symmetries of a part of the Langevin equation in order to derive properties of the full stochastic non-equilibrium model. To this end, we have compared the known exact results for the two-time autoresponse and autocorrelation functions in two specific models, see table 2, with the expressions derived from the standard field-theoretical actions which are habitually used to describe these systems. This is achieved through a decomposition of the action into two parts $S = S_0 + S_b$ such that (i) S_0 is Schrödinger invariant and the Bargman superselection rules hold for the averages calculated with S_0 only and (ii) the remaining terms contained in S_b are such that a perturbative expansion terminates at a finite order, again due to the Bargman superselection rules. The two models we considered, namely the bosonic variants of the critical-contact and pair-contact processes, satisfy these requirements and are clearly in agreement with the predictions of local scale invariance (LSI). In particular, our identification, equation (10), of the correct quasi-primary order parameter and response fields is likely to be useful in more general systems.

Specifically, we have seen the following.

(i) In the bosonic contact process, the symmetries of the noiseless part S_0 of the action is described in terms of the representation of the Schrödinger group relevant for the *free* diffusion equation.

In consequence, the form of the two-time response function is completely fixed by LSI and in agreement with the known exact result. The connected autocorrelator is exactly reducible to certain noiseless three- and four-point functions. Schrödinger invariance alone cannot determine these but the remaining free scaling functions can be chosen such that the known exact results can be reproduced.

(ii) For the bosonic pair-contact process, the symmetries of the partial action S_0 are described in terms of a new representation pertinent to a *nonlinear* Schrödinger equation. This new representation, which we have explicitly constructed, involves a dimensionful coupling constant g. Therefore even the response function is no longer fully determined. As for the autocorrelation function, which again can be exactly reduced to certain three- and four-point functions calculable from the action S_0, the remaining free scaling functions can be chosen as to fully reproduce the known exact results.

The consistency of the predictions of LSI with the exact results of these models furnishes further evidence in favour of an extension of the well-known dynamical scaling towards a (hidden) local scale invariance which influences the long-time behaviour of slowly relaxing systems. An essential ingredient was the Bargman superselection rules which at present can only be derived for a dynamical exponent $z = 2$. An extension of our method to models with $z \neq 2$ would first require a way to generalize the Bargman superselection rules. We hope to return to this open problem elsewhere.

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Appendix A. Representations of $\mathfrak{age}_1$ and $\mathfrak{sch}_1$ for semi-linear Schrödinger equations

We discuss the Schrödinger invariance of semi-linear Schrödinger equations of the form (57) and especially with nonlinearities of the form (58). With respect to the well-known Schrödinger invariance of the linear Schrödinger equation, the main difference comes from the presence of a dimensionful coupling constant g of the nonlinear term.

It is enough to consider explicitly the one-dimensional case which simplifies the notation. In one spatial dimension, the Schrödinger algebra $\mathfrak{sch}_1$ is spanned by the following generators:

$$\mathfrak{sch}_1 = \langle X_{-1}, X_0, X_1, Y_{-1/2}, Y_{1/2}, M_0 \rangle, \tag{A.1}$$

while its subalgebra $\mathfrak{age}_1$ is spanned by

$$\mathfrak{age}_1 = \langle X_0, X_1, Y_{-1/2}, Y_{1/2}, M_0 \rangle. \tag{A.2}$$

These generators for $g = 0$ are listed explicitly in equation (20) and the non-vanishing commutators can be written compactly:

$$\begin{aligned} [X_n, X_{n'}] &= (n - n')X_{n+n'} \\ [X_n, Y_m] &= (n/2 - m)Y_{n+m} \\ \left[Y_{\frac{1}{2}}, Y_{-\frac{1}{2}}\right] &= M_0, \end{aligned} \tag{A.3}$$

where $n, n' \in \{\pm 1, 0\}$ and $m \in \left\{\pm\frac{1}{2}\right\}$ (see [8] for generalizations to $d > 1$).

Following the procedure given in [17], we now construct new representations of $\mathfrak{age}_1$ and $\mathfrak{sch}_1$ which takes into account a dimensionful coupling g with scaling dimension $\hat{y}$ as follows.

(i) The generator of space translations reads simply

$$Y_{-\frac{1}{2}} = -\partial_r. \tag{A.4}$$

(ii) The generator of scaling transformations is assumed to take the form

$$X_0 = -t\partial_t - \frac{1}{2}r\partial_r - \hat{y}g\partial_g - \frac{x}{2}, \tag{A.5}$$

where $\hat{y}$ is the scaling dimension of the coupling g.

(iii) For $\mathfrak{sch}_1$ we also keep the usual generator of time translations

$$X_{-1} = -\partial_t. \tag{A.6}$$

(iv) The remaining generators we write in the most general form adding a possible g-dependence through yet unknown functions L, Q and P:

$$
\begin{aligned}
M_0 &= -\mathcal{M} - L(t,r,g)\partial_g \\
Y_{\frac{1}{2}} &= -t\partial_r - \mathcal{M}r - Q(t,r,g)\partial_g \\
X_1 &= -t^2\partial_t - tr\partial_r - \frac{\mathcal{M}}{2}r^2 - xt - P(t,r,g)\partial_g .
\end{aligned}
\tag{A.7}
$$

The representation given by equations (A.4)–(A.7) must satisfy the commutation relations (A.3) for $\mathfrak{age}_1$ or $\mathfrak{sch}_1$. From these conditions the undetermined functions L, Q and P are derived. A straightforward but slightly longish calculation along the lines of [17] shows that for $\mathfrak{age}_1$, one has

$$
L = 0, \qquad Q = 0, \qquad P = p_0(\mathcal{M})t^{\hat{y}+1}m(t/g). \tag{A.8}
$$

Here, $m(v)$ is an arbitrary differentiable function and $p_0(\mathcal{M})$ a $\mathcal{M}$-dependent constant. We shall use the shorthand $v = t^{\hat{y}}/g$ in what follows.

In consequence, for $\mathfrak{age}_1$ only the generator X_1 is modified with respect to the representation (20) and this is described by the function $m(v)$ and the constant $p_0(\mathcal{M})$.

On the other hand, for $\mathfrak{sch}_1$ the additional condition $[X_1, X_{-1}] = 2X_0$ leads to $p_0 = 2\hat{y}, m(v) = v^{-1}$.

Hence, the new representations are still given by equation (20) with the only exception of X_1 which reads

$$
\begin{aligned}
\mathfrak{age}_1\colon\quad X_1 &= -t^2\partial_t - tr\partial_r - p_0(\mathcal{M})t^{\hat{y}+1}m(t^{\hat{y}}/g)\partial_g - \frac{\mathcal{M}r^2}{2} - xt \\
\mathfrak{sch}_1\colon\quad X_1 &= -t^2\partial_t - tr\partial_r - 2\hat{y}tg\partial_g - \frac{\mathcal{M}r^2}{2} - xt .
\end{aligned}
\tag{A.9}
$$

We require in addition the invariance of linear Schrödinger equation $\left(2\mathcal{M}\partial_t - \partial_r^2\right)\phi = 0$ with respect to this new representation. In terms of the Schrödinger operator $\hat{S}$ this means

$$
[\hat{S}, \mathcal{X}] = \lambda\hat{S}, \qquad \text{where} \quad \hat{S} := 2M_0X_{-1} - Y_{-1/2}^2 \tag{A.10}
$$

and $\mathcal{X}$ is one of the generators of $\mathfrak{age}_1$ equation (A.2) or of $\mathfrak{sch}_1$ equation (A.1). Obviously, $\lambda = 0$ if $\mathcal{X} \in \langle X_{-1}, Y_{\pm 1/2}, M_0\rangle$ and $\lambda = -1$ if $\mathcal{X} = X_0$. Finally, for X_1 we have from the definition of the Schrödinger operator $\hat{S}$,

$$
\begin{aligned}
[\hat{S}, X_1] &= -4M_0X_0 + (Y_{1/2}Y_{-1/2} + Y_{-1/2}Y_{1/2}) \\
&= -2t\hat{S} + \mathcal{M}(1 - 2x - 4\hat{y}g\partial_g),
\end{aligned}
\tag{A.11}
$$

where in the second line the explicit forms (A.4)–(A.7) were used. This also holds for all those representations of $\mathfrak{age}_1$ for which there exists an operator $X_{-1} \notin \mathfrak{age}_1$ such that $[X_1, X_{-1}] = 2X_0$ and we shall restrict our attention to those in what follows. On the other hand, the direct calculation of the same commutator with the explicit form (A.9) gives for $\mathfrak{age}_1$

$$
[\hat{S}, X_1] = -2t\hat{S} + \mathcal{M}(1 - 2x) - 2\mathcal{M}p_0(\mathcal{M})t^{\hat{y}}[(\hat{y}+1)m(v) + \hat{y}vm'(v))]\partial_g . \tag{A.12}
$$

Besides $\lambda = -2t$, the consistency between these two implies for $m(v)$ the equation

$$
v\left((\hat{y}+1)m(v) + \hat{y}v\frac{\mathrm{d}m(v)}{\mathrm{d}v}\right) = \frac{2\hat{y}}{p_0} \tag{A.13}
$$

with the general solution

$$
m(v) = \frac{2\hat{y}}{p_0}\frac{1}{v} + \frac{m_0}{p_0}v^{-1-1/\hat{y}}, \tag{A.14}
$$

where $m_0 = m_0(\mathcal{M})$ is an arbitrary constant. The larger algebra $\mathfrak{sch}_1$ is recovered from this if we set $p_0 = 2\hat{y}$ and $m_0 = 0$. Hence the final form for the generator X_1 in the special class of representations of the algebra $\mathfrak{age}_1$ defined above is

$$X_1 = -t^2\partial_t - tr\partial_r - 2\hat{y}tg\partial_g - m_0 g^{1+1/\hat{y}}\partial_g - \frac{\mathcal{M}r^2}{2} - xt. \tag{A.15}$$

Summarizing, this class of representations of $\mathfrak{age}_1$ we constructed is characterized by the triplet $(x, \mathcal{M}, m_0)$, whereas for $\mathfrak{sch}_1$, the same triplet is $(x, \mathcal{M}, 0)$.

Finally, to make X_1 a dynamical symmetry on the solutions $\Phi = \Phi_g(t, r)$ of the Schrödinger equation $\hat{S}\Phi_g = 0$ we must impose the auxiliary condition $(1-2x-4\hat{y}g\partial_g)\Phi_g = 0$ which leads to

$$\Phi_g(t, r) = g^{(1-2x)/(4\hat{y})}\Phi(t, r). \tag{A.16}$$

In particular, we see that if $x = 1/2$, we have a representation of $\mathfrak{age}_1$ without any further auxiliary condition.

We now look for those semi-linear Schrödinger equations of the form $\hat{S}\Phi = F(t, r, g, \Phi, \Phi^*)$ for which the representations of $\mathfrak{age}_1$ or $\mathfrak{sch}_1$ as given by equations (A.4)–(A.7) and with X_1 as in (A.15) act as a dynamical symmetry. The nonlinear potential F is known to satisfy certain differential equations which can be found using standard methods, see [33], [17, equation (2.8)]. In our case these equations read

$$X_{-1}:\quad \partial_t F = 0 \tag{A.17}$$

$$Y_{-\frac{1}{2}}:\quad \partial_r F = 0 \tag{A.18}$$

$$M_0:\quad (\Phi\partial_\Phi - \Phi^*\partial_{\Phi^*} - 1)F = 0 \tag{A.19}$$

$$Y_{\frac{1}{2}}:\quad [t\partial_r F - \mathcal{M}r(\Phi\partial_\Phi - \Phi^*\partial_{\Phi^*} - 1)]F = 0 \tag{A.20}$$

$$X_0:\quad \left[t\partial_t + \frac{1}{2}r\partial_r + \hat{y}g\partial_g + 1 - \frac{x}{2}(\Phi\partial_\Phi + \Phi^*\partial_{\Phi^*} - 1)\right]F = 0 \tag{A.21}$$

$$X_1:\quad \left[t^2\partial_t + tr\partial_r + 2t(\hat{y}g\partial_g + 1) + m_0 g^{1+1/\hat{y}}\partial_g \right.$$
$$\left. - \frac{\mathcal{M}r^2}{2}(\Phi\partial_\Phi - \Phi^*\partial_{\Phi^*} - 1) - xt(\Phi\partial_\Phi + \Phi^*\partial_{\Phi^*} - 1)\right]F = 0. \tag{A.22}$$

We first solve these for $\mathfrak{sch}_1$. From conditions (A.17)–(A.21) we easily find

$$F = \Phi(\Phi\Phi^*)^{1/x} f(g^x(\Phi\Phi^*)^{\hat{y}}), \tag{A.23}$$

where f is an arbitrary differentiable function. Two comments are in order:

(i) For a dimensionless coupling g, that is $\hat{y} = 0$, we have $x = 1/2$. Then the scaling function reduces to a g-dependent constant and we recover the standard form for the nonlinear potential F as quoted ubiquitously in the mathematical literature, see e.g. [34].

(ii) Taking into account the generator X_1 from equation (A.22) as well does not change the result. Hence in this case translation-, dilatation- and Galilei-invariance are indeed sufficient for the special Schrödinger invariance generated by X_1, see also [31]. We point out that traditionally an analogous assertion holds for conformal field theory, see e.g. [1], but counterexamples are known where in local theories scale- and translation-invariance are *not* sufficient for conformal invariance [35, 36].

Second, we now consider the representation of $\mathfrak{age}_1$ where X_1 is given by (A.15). We have conditions (A.18)–(A.22). We write $F = \Phi\mathcal{F}(u, t, g)$ with $u = \Phi\Phi^*$ and the remaining equations coming from X_0 and X_1 are

$$(t\partial_t + \hat{y}g\partial_g - xu\partial_u + 1)\mathcal{F} = 0 \qquad (t^2\partial_t + m_0 g^{1+1/\hat{y}}\partial_g)\mathcal{F} = 0 \tag{A.24}$$

with the final result

$$F = \Phi(\Phi\Phi^*)^{1/x} f\left((\Phi\Phi^*)^{\hat{y}}\left[g^{-1/\hat{y}} - \frac{m_0}{\hat{y}t}\right]^{-x\hat{y}}\right), \tag{A.25}$$

where f is the *same* scaling function as encountered before for $\mathfrak{sch}_1$. Finally, the results for the general representations of $\mathfrak{age}_1$ which depend on an arbitrary function $m(v)$ are not particularly inspiring and will not be detailed here. We observe

(i) For $m_0 = 0$, this result is identical to that found for $\mathfrak{sch}_1$;
(ii) Even for $m_0 \neq 0$, the form of the nonlinear potential reduces in the long-time limit $t \to \infty$ to that found in equation (A.23) for the larger algebra $\mathfrak{sch}_1$.

We can summarize the main results of this appendix as follows.

Proposition. *Consider the following generators,*

$$\begin{aligned}
&M_0 = -\mathcal{M}, \qquad Y_{-1/2} = -\partial_r, \qquad Y_{1/2} = -t\partial_r - \mathcal{M}r, \qquad X_{-1} = -\partial_t \\
&X_0 = -t\partial_t - \frac{1}{2}r\partial_r - \hat{y}g\partial_g - \frac{x}{2} \\
&X_1 = -t^2\partial_t - tr\partial_r - 2\hat{y}tg\partial_g - m_0 g^{1+1/\hat{y}}\partial_g - \frac{\mathcal{M}r^2}{2} - xt,
\end{aligned} \tag{A.26}$$

where $x, \mathcal{M}, m_0$ are parameters. Define the Schrödinger operator $\hat{S} := 2M_0X_{-1} - Y_{-1/2}^2$. Then

(i) the generators $\langle X_{0,1}, Y_{\pm 1/2}, M_0\rangle$ form a representation of the Lie algebra $\mathfrak{age}_1$. If furthermore $m_0 = 0$, then $\langle X_{0,\pm 1}, Y_{\pm 1/2}, M_0\rangle$ is a representation of the Lie algebra $\mathfrak{sch}_1$.
(ii) These representations are dynamical symmetries of the Schrödinger equation $\hat{S}\Phi = 0$, under the auxiliary condition $(1 - 2x - 4\hat{y}g\partial_g)\Phi = 0$.
(iii) For the Schrödinger algebra $\mathfrak{sch}_1$ and also in the asymptotic limit $t \to \infty$ for the ageing algebra $\mathfrak{age}_1$, the semi-linear Schrödinger equation invariant under these representations has the form

$$\hat{S}\Phi = \Phi(\Phi\Phi^*)^{1/x} f(g^x(\Phi\Phi^*)^{\hat{y}}), \tag{A.27}$$

where f is an arbitrary differentiable function.

This general form includes our potential (58) since the scaling dimension $\hat{y}$ is a remaining free parameter in our considerations.

Appendix B. The *n*-point function

In this appendix we use the generators (A.26) from appendix A to find the most general form of the n-point functions compatible with the symmetries for $n \geqslant 3$. We shall do this for the case $m_0 = 0$ only, as this will be enough to reproduce the exact results of table 2. The case $n = 2$ needs a special treatment and is presented in appendix C.

We restrict ourselves to the case $d = 1$ for simplicity, but the generalization to arbitrary dimension will be obvious. First we introduce some notation. We fix an arbitrary index k and define the shifted coordinates

$$\tilde{\boldsymbol{r}}_b := \boldsymbol{r}_b - \boldsymbol{r}_k, \qquad \tilde{t}_b := t_b - t_k \qquad \text{for} \qquad j \neq k \quad \text{and} \quad \tilde{t}_k := t_k. \tag{B.1}$$

In the following, we will adopt the following convention: the index a always runs from 1 to n, the index b runs from 1 to n but skips k. The prime on a sum means that the index k is left out, namely,

$$\sum_{i=1}^{n}{}' A_i := \sum_{\substack{i=1\\ i\neq k}}^{n} A_i. \tag{B.2}$$

We denote the n-point function by

$$F(\{\boldsymbol{r}_a\}, \{t_a\}, \{g_a\}) := \langle \varphi_1(\boldsymbol{r}_1, t_1) \ldots \varphi_n(r_n, t_n)\rangle, \tag{B.3}$$

where we assume one coupling constant for each field. This quantity has to satisfy the following four linear partial differential equations:

$$\left(\sum_{i=1}^{n} X_k^{(i)}\right) F(\{\boldsymbol{r}_a\}, \{t_a\}, \{t_a\}) = 0, \qquad k \in \{0, 1\} \tag{B.4}$$

$$\left(\sum_{i=1}^{n} Y_m^{(i)}\right) F(\{\boldsymbol{r}_a\}, \{t_a\}, \{t_a\}) = 0, \qquad m \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}. \tag{B.5}$$

To solve these equations, we use the method of characteristics [37]. We solve (B.4) first for spatial translation invariance with the result

$$F(\{\boldsymbol{r}_a\}, \{t_a\}, \{g_a\}) = \tilde{F}(\{\tilde{\boldsymbol{r}}_b\}, \{t_a\}, \{g_a\}) \tag{B.6}$$

with a new function $\tilde{F}$ with $3n - 1$ arguments. In order to solve for X_0 we set

$$x = \frac{1}{2}\sum_{i=1}^{n} x_i \tag{B.7}$$

and make the ansatz

$$\tilde{F}(\{\tilde{\boldsymbol{r}}_b\}, \{t_a\}, \{g_a\}) = \prod_{i<j} (t_i - t_j)^{-\rho_{ij}} G(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\}), \tag{B.8}$$

where the parameters ρ_{ij} and the function G remain to be determined. We also change to the new independent temporal variables $\tilde{t}_a$. Then one finds after a short calculation

$$\left(\sum_{i=1}^{n} \tilde{t}_i \partial_{\tilde{t}_i} + \frac{1}{2}\sum_{i=1}^{n}{}' \tilde{\boldsymbol{r}}_i \partial_{\tilde{\boldsymbol{r}}_i} + \sum_{i=1}^{n} \hat{y}_i g_i \partial_{g_i}\right) G(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0 \tag{B.9}$$

together with the condition

$$x = \sum_{i<j} \rho_{ij}. \tag{B.10}$$

Before proceeding to solve this equation, we turn to the generators $Y_{1/2}$. We find for the function $G(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\})$,

$$\left(\sum_{i=1}^{n}{}' \tilde{t}_i \partial_{\tilde{\boldsymbol{r}}_i} + \sum_{i=1}^{n}{}' \mathcal{M}_i \tilde{\boldsymbol{r}}_i + \boldsymbol{r}_k \left(\sum_{i=1}^{n} \mathcal{M}_i\right)\right) G(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0. \tag{B.11}$$

Since $G(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\})$ does not depend on $\boldsymbol{r}_k$, we recover the Bargman superselection rule

$$\sum_{i=1}^{n} \mathcal{M}_i = 0 \tag{B.12}$$

as expected. For $G(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\})$ we make another ansatz:

$$G(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\}) = \exp\left(-\sum_{i=1}^{n}{}' \frac{\mathcal{M}_i}{2} \frac{\tilde{\boldsymbol{r}}_i^2}{\tilde{t}_i}\right) H(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\}), \tag{B.13}$$

where the function $H(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\})$ remains to be determined. With (B.12) and (B.13), equation (B.11) reduces to

$$\left(\sum_{i=1}^{n}{}' \tilde{t}_i \partial_{\tilde{\boldsymbol{r}}_i}\right) H(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0. \tag{B.14}$$

We retake (B.9) and introduce the ansatz (B.13). This yields

$$\left(\sum_{i=1}^{n} \tilde{t}_i \partial_{\tilde{t}_i} + \frac{1}{2}\sum_{i=1}^{n}{}' \tilde{\boldsymbol{r}}_i \partial_{\tilde{\boldsymbol{r}}_i} + \sum_{i=1}^{n} \hat{y}_i g_i \partial_{g_i}\right) H(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0. \tag{B.15}$$

The last equation for $H(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\})$ we obtain from X_1. Using the ansatz (B.8) yields an equation for G,

$$\begin{aligned}\Bigg(\sum_{i=1}^{n} \tilde{t}_i^2 \partial_{\tilde{t}_i} + \sum_{i=1}^{n}{}' \tilde{t}_i \tilde{\boldsymbol{r}}_i \partial_{\tilde{\boldsymbol{r}}_i} + \frac{1}{2}\sum_{i=1}^{n}{}' \mathcal{M}_i \tilde{\boldsymbol{r}}_i^2 + 2\sum_{i=1}^{n}{}' \hat{y}_i \tilde{t}_i g_i \partial_{g_i} + \boldsymbol{r}_k \left(\sum_{i=1}^{n}{}' \tilde{t}_i \partial_{\tilde{\boldsymbol{r}}_i} + \sum_{i=1}^{n}{}' \tilde{\boldsymbol{r}}_i \mathcal{M}_i\right) \\ + 2\tilde{t}_k \left(\sum_{i=1}^{n}{}' \tilde{t}_i \partial_{\tilde{t}_i} + \frac{1}{2}\sum_{i=1}^{n}{}' \tilde{\boldsymbol{r}}_i \partial_{\boldsymbol{r}_i} + \sum_{i=1}^{n}{}' \hat{y}_i g_i \partial_{g_i}\right)\Bigg) G(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0.\end{aligned} \tag{B.16}$$

Together with the condition

$$\sum_{i<j} \rho_{ij}(t_i + t_j) = \sum_{i=1}^{n} t_i x_i, \tag{B.17}$$

which is satisfied if we choose the parameters ρ_{ij} such that

$$\begin{aligned} x_1 &= \rho_{12} + \rho_{13} + \rho_{14} + \rho_{15} + \cdots + \rho_{1n} \\ x_2 &= \rho_{12} + \rho_{23} + \rho_{24} + \rho_{25} + \cdots + \rho_{2n} \\ x_3 &= \rho_{13} + \rho_{23} + \rho_{34} + \rho_{35} + \cdots + \rho_{3n} \\ &\vdots \\ x_n &= \rho_{1n} + \rho_{2n} + \rho_{3n} + \rho_{4n} + \cdots + \rho_{n-1n}. \end{aligned} \tag{B.18}$$

Here a few remarks are in order. The above system is compatible with (B.10), as can be seen by adding all equations. Also, this system is always solvable for $n \geqslant 3$, as for $n \geqslant 4$, it is underdetermined and for $n = 3$ the corresponding determinant does not vanish[14]. Lastly, we often have the case $x_1 = x_2 =: x$ and $x_3 = \cdots = x_n =: \tilde{x}$. In this case, we can set

$$\rho_{12} = x - \frac{n-2}{2}\tilde{x}; \qquad \rho_{2i} = \frac{1}{2}\tilde{x}, \quad \rho_{1i} = \frac{1}{2}\tilde{x} \quad \text{for} \quad i = 3, \ldots, n \tag{B.19}$$

[14] This system is not solvable for $n = 2$ when $x_1 \neq x_2$. This case is considered in appendix C.

and $\rho_{ij} = 0$ for all the remaining ρ_{ij}. We still have to rewrite equation (B.16) in terms of the variables $\{\tilde{\boldsymbol{r}}_b\}$ and $\{\tilde{t}_a\}$. Here we take equations (B.9) and (B.11) and the ansatz (B.13) into account and get for $H(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\})$,

$$\left(\sum_{i=1}^{n}{}' \tilde{t}_i^2 \partial_{\tilde{t}_i} - \tilde{t}_k^2 \partial_{t_k} + \sum_{i=1}^{n}{}' \tilde{t}_i \tilde{\boldsymbol{r}}_i \partial_{\tilde{\boldsymbol{r}}_i} + 2 \sum_{i=1}^{n}{}' \hat{y}_i \tilde{t}_i g_i \partial_{g_i}\right) H(\{\tilde{\boldsymbol{r}}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0. \tag{B.20}$$

We thus have to solve the homogenous equations (B.14), (B.15) and (B.20). This will eliminate three more variables and yields

$$F(\{\boldsymbol{r}_a\}, \{t_a\}, \{g_a\}) = \prod_{i<j} (t_i - t_j)^{-\rho_{ij}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n}{}' \mathcal{M}_i \frac{(\boldsymbol{r}_i - \boldsymbol{r}_k)^2}{t_i - t_k}\right) \tilde{\Psi}_n(\{u_c\}, \{v_c\}, \{\beta_a\}) \tag{B.21}$$

with an arbitrary function $\tilde{\Psi}_n$, which depends on $3n - 4$ variables. Here the index c runs from 1 to n but skips k and another arbitrarily fixed index $r \neq k$, and the expressions u_c, v_c and β_a are given by

$$\begin{aligned} u_c &= \frac{t_k\,((\boldsymbol{r}_c - \boldsymbol{r}_k)(t_r - t_k) - (\boldsymbol{r}_r - \boldsymbol{r}_k)(t_c - t_k))^2}{(t_c - t_k)(t_r - t_k)^2 t_c} \\ v_c &= \frac{t_k\,((\boldsymbol{r}_c - \boldsymbol{r}_k)(t_r - t_k) - (\boldsymbol{r}_r - \boldsymbol{r}_k)(t_c - t_k))^2}{(t_r - t_k)(t_c - t_k)^2 t_r} \\ \beta_k &= g_k{}^{(1/\hat{y}_k)} \left(\frac{t_r}{(t_r - t_k) t_k}\right), \qquad \beta_b = g_b{}^{(1/\hat{y}_b)} \left(\frac{t_k (t_b - t_k)^2}{(t_r - t_k) t_r}\right). \end{aligned} \tag{B.22}$$

We remind the reader of our convention that the index c runs from 1 to n skipping r and k and that the index b runs from 1 to n skipping only k.

In higher dimensions rotational invariance has to be satisfied as well and then the generalization to arbitrary d is straightforward.

If we consider instead the algebra $\mathfrak{age}_1$ with dimensionless couplings g_i we merely have to make the replacement

$$\tilde{\Psi}_n(\{u_c\}, \{v_c\}, \{\beta_a\}) \longrightarrow \Psi_n(\{u_c\}, \{v_c\}), \tag{B.23}$$

where Ψ_n is also an arbitrary function such that only the dependence on $\{\beta_a\}$ drops out.

Finally, we explicitly list the three- and four-point functions in the form in which they are needed in the main text. The three-point function with fixed indices $r = 2$ and $k = 3$ and the special situation (B.19) assumed reads

$$\begin{aligned} F(\{\boldsymbol{r}_a\}, \{t_a\}, \{g_a\}) &= (t_1 - t_2)^{-(x - \frac{1}{2}\tilde{x})} (t_1 - t_3)^{-\frac{1}{2}\tilde{x}} (t_2 - t_3)^{-\frac{1}{2}\tilde{x}} \\ &\quad \times \exp\left(-\frac{1}{2} \sum_{i=1}^{2} \mathcal{M}_i \frac{(\boldsymbol{r}_i - \boldsymbol{r}_3)^2}{t_i - t_3}\right) \tilde{\Psi}_n(\{u_c\}, \{v_c\}, \{\beta_a\}) \end{aligned} \tag{B.24}$$

with

$$\begin{aligned} u_1 &= \frac{t_3}{t_1} \frac{[(\boldsymbol{r}_1 - \boldsymbol{r}_3)(t_2 - t_3) - (\boldsymbol{r}_2 - \boldsymbol{r}_3)(t_1 - t_3)]^2}{(t_1 - t_3)(t_2 - t_3)^2} \\ v_1 &= \frac{t_3}{t_2} \frac{[(\boldsymbol{r}_1 - \boldsymbol{r}_3)(t_2 - t_3) - (\boldsymbol{r}_2 - \boldsymbol{r}_3)(t_1 - t_3)]^2}{(t_1 - t_3)^2 (t_2 - t_3)} \\ \beta_1 &= g_1{}^{1/\hat{y}_1} \frac{t_3 (t_1 - t_3)^2}{(t_2 - t_3) t_2}, \qquad \beta_2 = g_2{}^{1/\hat{y}_2} \frac{t_3 (t_2 - t_3)}{t_2} \\ \beta_3 &= g_3{}^{1/\hat{y}_3} \frac{t_2}{t_3 (t_2 - t_3)}. \end{aligned} \tag{B.25}$$

The four-point function with $r = 1$ and $k = 2$ in the special situation (B.19) reads

$$F(\{\boldsymbol{r}_a\}, \{t_a\}, \{g_a\}) = (t_1 - t_2)^{-(x-\tilde{x})}(t_1 - t_3)^{-\frac{1}{2}\tilde{x}}(t_1 - t_4)^{-\frac{1}{2}\tilde{x}}(t_2 - t_3)^{-\frac{1}{2}\tilde{x}} \times (t_2 - t_4)^{-\frac{1}{2}\tilde{x}} \exp\left(-\frac{1}{2}\sum_{i=1}^{3}\mathcal{M}_i \frac{(\boldsymbol{r}_i - \boldsymbol{r}_4)^2}{t_i - t_4}\right) \tilde{\Psi}_n(\{u_c\}, \{v_c\}, \{\beta_a\})$$

with

$$\begin{aligned} u_3 &= \frac{t_2}{t_3}\frac{[(\boldsymbol{r}_3 - \boldsymbol{r}_2)(t_1 - t_2) - (\boldsymbol{r}_1 - \boldsymbol{r}_2)(t_3 - t_4)]^2}{(t_3 - t_2)(t_1 - t_2)^2} \\ u_4 &= \frac{t_2}{t_4}\frac{[(\boldsymbol{r}_4 - \boldsymbol{r}_2)(t_1 - t_2) - (\boldsymbol{r}_1 - \boldsymbol{r}_2)(t_4 - t_2)]^2}{(t_4 - t_2)(t_1 - t_2)^2} \\ v_3 &= \frac{t_2}{t_1}\frac{[(\boldsymbol{r}_3 - \boldsymbol{r}_2)(t_1 - t_2) - (\boldsymbol{r}_1 - \boldsymbol{r}_2)(t_3 - t_2)]^2}{(t_1 - t_2)(t_3 - t_2)^2} \\ v_4 &= \frac{t_2}{t_1}\frac{[(\boldsymbol{r}_4 - \boldsymbol{r}_2)(t_1 - t_2) - (\boldsymbol{r}_1 - \boldsymbol{r}_2)(t_4 - t_2)]^2}{(t_1 - t_2)(t_4 - t_2)^2} \\ \beta_1 &= g_1{}^{1/\hat{y}_1}\frac{t_2(t_1 - t_2)}{t_1}, \qquad \beta_2 = g_2{}^{1/\hat{y}_2}\frac{t_1}{(t_1 - t_2)t_2} \\ \beta_3 &= g_3{}^{1/\hat{y}_3}\frac{t_2(t_3 - t_2)^2}{t_1(t_1 - t_2)}, \qquad \beta_4 = g_4{}^{1/\hat{y}_4}\frac{t_2(t_4 - t_2)^2}{(t_1 - t_2)t_1}. \end{aligned} \tag{B.26}$$

Appendix C. The two-point function

In this appendix we calculate the two-point function, which was not included in the treatment of appendix B. Again, we only treat the case $m_0 = 0$. Apart from the generator X_1, the calculations are similar to those done in appendix B, so we only give the essential steps. First we define

$$\tau := t_1 - t_2, \qquad \boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{C.1}$$

and then we proceed as follows. We solve for $M_0, Y_{-1/2}, Y_{1/2}, X_0$ in exactly the same way as before with the result

$$F(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1, t_2, g_1, g_2) = \langle\varphi_1(\boldsymbol{r}_1, t_1, g_1)\varphi_2(\boldsymbol{r}_2, t_2, g_2)\rangle_0 = \tau^{-x}G(\boldsymbol{r}, \tau, t_2, g_1, g_2), \tag{C.2}$$

where $x = \frac{1}{2}(x_1 + x_2)$ and $G(\boldsymbol{r}, \tau, t_2, g_1, g_2)$ satisfies the equations

$$\left(\tau\partial_\tau + t_2\partial_{t_2} + \tfrac{1}{2}\boldsymbol{r}\partial_{\boldsymbol{r}} + y_1g_1\partial_{g_1} + y_2g_2\partial_{g_2}\right)G(\boldsymbol{r}, \tau, t_2, g_1, g_2) = 0 \tag{C.3}$$

$$(\tau\partial_{\boldsymbol{r}} + \boldsymbol{r}\mathcal{M}_1)G(\boldsymbol{r}, \tau, t_2, g_1, g_2) = 0 \tag{C.4}$$

and the Bargman superselection rule

$$\mathcal{M}_1 + \mathcal{M}_2 = 0 \tag{C.5}$$

holds true. Now (C.3) is solved by

$$G(\boldsymbol{r}, \tau, t_2, g_1, g_2) = \tilde{G}(u_1, u_2, v_1, v_2), \tag{C.6}$$

where we have defined

$$u_1 := \frac{\boldsymbol{r}^2}{\tau}, \qquad u_2 := \frac{\boldsymbol{r}^2}{t_2}, \qquad v_1 := \frac{g_1^{1/\hat{y}_1}}{\tau}, \qquad v_2 := \frac{g_2^{1/\hat{y}_2}}{\tau} \tag{C.7}$$

and rewriting (C.4) in terms of the new variables yields

$$\left(u_1\partial_{u_1} + u_2\partial_{u_2} + \frac{1}{2}u_1\mathcal{M}_1\right)\tilde{G}(u_1, u_2, v_1, v_2) = 0, \tag{C.8}$$

which is solved by

$$\tilde{G}(u_1, u_2, v_1, v_2) = \exp\left(-\frac{1}{2}u_1\mathcal{M}_1\right)H(w, v_1, v_2), \qquad w := \frac{u_2}{u_1}. \tag{C.9}$$

The function $H(w, v_1, v_2)$ is found through the generator X_1. Using again the invariance under $Y_{1/2}$ and X_0, we readily obtain in terms of v_1, v_2 and w

$$\left((w+1)\partial_w + v_1\partial_{v_1} - v_2\partial_{v_2} + \frac{1}{2}(x_1 - x_2)\right)H(w, v_1, v_2) = 0. \tag{C.10}$$

The most general solution of this equation is

$$H = (w+1)^{-\frac{1}{2}(x_1-x_2)}\tilde{\Psi}_2\left(\frac{(w+1)}{v_1}, v_1v_2\right), \tag{C.11}$$

where the function $\tilde{\Psi}_2$ remains arbitrary. Substituting back the values for v_1, v_2 and w our final result is

$$F(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2) = \delta_{\mathcal{M}_1+\mathcal{M}_2,0}(t_1 - t_2)^{-\frac{1}{2}(x_1+x_2)}\left(\frac{t_1}{t_2}\right)^{-\frac{1}{2}(x_1-x_2)}$$
$$\times \exp\left(-\frac{\mathcal{M}_1}{2}\frac{(\boldsymbol{r}_1-\boldsymbol{r}_2)^2}{t_1-t_2}\right)\tilde{\Psi}_2\left(\left(\frac{t_1}{t_2}\right)^{\hat{y}_1}\frac{(t_1-t_2)^{\hat{y}_1}}{g_1}, \frac{g_1g_2}{(t_1-t_2)^{\hat{y}_1+\hat{y}_2}}\right). \tag{C.12}$$

For applications to semi-linear equations, one now sets $g := g_1 = g_2$ with a scaling dimension $\hat{y} := \hat{y}_1 = \hat{y}_2$. In the limit $\hat{y} \to 0$, the function $\tilde{\Psi}_2$ reduces to a g-dependent normalization constant and we recover the standard result [29].

In many applications, one expects the scaling functions to be universal, up to normalization. On the other hand, the coupling g should be a non-universal quantity so that a universal scaling function cannot contain g in its arguments. This leads to $\tilde{\Psi}_2 = \tilde{\Psi}_2((t_1/t_2)^{\hat{y}})$ and we point out that such a scaling form would be compatible (one still has $z = 2$, however) with what is found from the field-theoretical renormalization group and numerical simulations in non-equilibrium critical dynamics [5, 13]. An extension to different values of z would as a first step require the generalization of the Bargman superselection rules. We hope to come back elsewhere to this open problem.

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